Basic Electromagnetic Theory

Maxwell's Equations:

Maxwell's equations were initially postulated based on experimental evidence, and have since been found to govern all classical electromagnetic phenomena. In both integral and differential form, they are written as:

$\nabla \cdot \overline{D} = \rho$	$\oint_S \overline{D} \cdot d\overline{S} = \int_V \rho \, dV$	Gauss' Law
$\nabla \cdot \overline{B} = 0$	$\oint_S \overline{B} \cdot d\overline{S} = 0$	No Magnetic Charges
$\nabla \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t}$	$\oint_C \overline{E} \cdot d\ell = -\dfrac{\partial}{\partial t} \int_S \overline{B} \cdot d\overline{S}$	Faraday's Law
$\nabla \times \overline{H} = \overline{J} + \dfrac{\partial \overline{D}}{\partial t}$	$\oint_C \overline{H} \cdot d\ell = \int_S \overline{J} \cdot d\overline{S} + \dfrac{\partial}{\partial t} \int_S \overline{D} \cdot d\overline{S}$	Modified Ampere's Law

where
$\overline{E} =$ Electric Field Intensity [V/m]
$\overline{H} =$ Magnetic Field Intensity [A/m]
$\overline{J} =$ Electric Current Density [A/m^2]
$\overline{D} =$ Electric Flux Density [C/m^2]
$\overline{B} =$ Magnetic Flux Density [T]
$\rho =$ Electric Charge Density [C/m^3]

These equations are consistent with the conservation of charge, which is represented by the continuity equation for current:

$$\nabla \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0$$

Constitutive Relations:

The two divergence equations can be derived from the curl equations and the continuity equation. Hence Maxwell's equations represent six scalar equations with twelve unknowns. The remaining six scalar equations required for a unique field solution are found using the constitutive equations, which relate the fields in a certain material as:

$$\overline{D} = \overline{\overline{\epsilon}}\,\overline{E} \qquad \overline{B} = \overline{\overline{\mu}}\,\overline{H}$$

where $\overline{\overline{\epsilon}}$ is the permittivity and $\overline{\overline{\mu}}$ is the permeability of the material, and both are tensors in general. In simple media, ϵ and μ are scalars, and

$$\overline{D} = \epsilon\overline{E} = \epsilon_0\epsilon_r\overline{E} \qquad \overline{B} = \mu\overline{H} = \mu_0\mu_r\overline{H}$$

where ϵ_r and μ_r are the relative permittivity and relative permeability of the medium.

Boundary Conditions:

At boundaries abruptly separating two dissimilar materials, the following conditions hold:

$$\hat{n} \times (\overline{E}_1 - \overline{E}_2) = 0 \qquad \hat{n} \cdot (\overline{D}_1 - \overline{D}_2) = \rho_s$$
$$\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s \qquad \hat{n} \cdot (\overline{B}_1 - \overline{B}_2) = 0$$

These are derived using the integral form of Maxwell's equations. In this form, the normal vector $\hat{n}$ to the boundary points from region-2 to region-1. $\overline{J}_s$ is a surface current density, with units of [A/m], and ρ_s is a surface charge density [C/m^2].

Charges in Motion:

A charge which moves with velocity $\overline{v}$ in an electric and magnetic field will experience a force, which can be calculated from:

$$\text{Lorentz Force Law} \qquad \overline{F} = q\left(\overline{E} + \overline{v} \times \overline{B}\right)$$

This expression is actually considered as the *definition* of the electric and magnetic fields. Charges in motion also give rise to a current density, which is given by

$$\text{Convection Current} \qquad \overline{J} = \rho \overline{v}$$

This expression applies to charges moving through conducting materials as well. In that case, however, the model is complicated by scattering processes in the material. Experimentally, it has been found that the net drift velocity of the charges is proportional to the applied field in the conductor, which is expressed through:

$$\text{Ohm's Law for Conduction Current} \qquad \overline{J} = \sigma \overline{E}$$

where σ is defined as the conductivity [S/m] of the material. It can be shown that the assumption of Ohm's law implies the absence of a *net* volume charge density in the steady-state. For poor conductors, the time to reach the this equilibrium is on the order of $\tau_d = \epsilon/\sigma$, the dielectric relaxation time. For good conductors, the mean-time between collisions is more appropriate. The total current is related to the current density by

$$I = \int_S \overline{J} \cdot d\overline{S}$$

The DC resistance of a uniform bar of conducting material is given by

$$R = \frac{1}{\sigma}\frac{L}{A}$$

where L is the length of the bar, and A is the cross-sectional area.

Common Terminology:

1.) A homogeneous medium is one in which the material constants are uniform everywhere in the region of interest. An inhomogeneous material varies in composition, so that the material constants are functions of position, such as $\epsilon(x)$ and $\mu(x)$.

2.) In an isotropic material, the material constants *do not* depend on the polarization of the electromagnetic field. In an anisotropic medium, the material constants *are* functions of the field orientation, and hence must be written as tensors, or matrices, $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$.

3.) In linear materials, the constitutive relations are, well...linear! That is, they do not involve powers or other nonlinear functions of the field variables. The opposite is a nonlinear material, in which the material constants are functions of the applied field strength, such as $\epsilon(E)$ and $\mu(H)$. In general, most materials do exhibit some nonlinearity at very high field strengths, but this regime is usually avoided.

4.) Not surprisingly, in time-invariant media the material constants do not vary with time. On the other hand, time-varying media may have $\epsilon(t)$ or $\mu(t)$, for example.

5.) The charge density and current density are considered the sources of electric and magnetic fields. Hence a source-free region is where $(\rho = J = 0)$.

6.) Materials in which the material constants are functions of frequency, $\epsilon(\omega)$ or $\mu(\omega)$, are called dispersive. In dispersive media, the wave velocity will be a function of frequency, which means that modulated or pulsed signals will spread, or disperse, as they propagate. This leads to a definition of group velocity, the velocity of energy or information travel, which must necessarily consist of some collection of frequencies.

7.) When a material is said to be non-magnetic, this means that $\mu = \mu_0$.

Energy, Power, and Poynting's Theorem:

In electromagnetics, the Poynting vector $\overline{\mathcal{P}}$ is defined as

$$\text{Poynting vector:} \qquad \overline{\mathcal{P}} \equiv \overline{E} \times \overline{H} \quad [\text{W/m}^2]$$

and is associated with the power *density* carried by the electric and magnetic fields. The direction of $\overline{\mathcal{P}}$ indicates the direction of power flow. Using Maxwell's equations, the flux, or flow of the Poynting vector out of some closed surface S can be expressed as

$$\text{Poynting's theorem:} \qquad \oint_S (\overline{E} \times \overline{H}) \cdot d\overline{S} = -\int_V \left[\frac{\partial}{\partial t} \left(\frac{\overline{B} \cdot \overline{H}}{2} \right) + \frac{\partial}{\partial t} \left(\frac{\overline{D} \cdot \overline{E}}{2} \right) + \overline{E} \cdot \overline{J} \right] dV$$

This can be interpreted by identifying three forms of energy:

$$\text{Stored Electric Energy Density:} \quad w_e = \frac{1}{2} \overline{D} \cdot \overline{E} = \frac{1}{2} \epsilon |E|^2$$

$$\text{Stored Magnetic Energy Density:} \quad w_m = \frac{1}{2} \overline{B} \cdot \overline{H} = \frac{1}{2} \mu |H|^2$$

$$\text{Ohmic Losses (heat):} \quad p_\sigma = \overline{E} \cdot \overline{J} = \sigma |E|^2$$

Poynting's theorem then becomes

$$-\oint_S \overline{\mathcal{P}} \cdot d\overline{S} = \frac{\partial}{\partial t} \int_V (w_e + w_m)\, dV + \int_V p_\sigma \, dV$$

which says that the flow of power *into* some closed surface S equals the rate of *increase* of the stored energy within that volume, plus any ohmic power loss within that region. For sinusoidally varying fields, the Poynting vector can be expressed in terms of phasors as

$$\text{complex Poynting vector:} \qquad \overline{\mathcal{P}} = \overline{E} \times \overline{H}^*$$

This gives the *instantaneous* power density, however the *average* power density is usually more important, and this is found to be

$$\text{Time-averaged Power Density:} \qquad \overline{\mathcal{P}}_{\text{ave}} = \frac{1}{2} \text{Re} \left\{ \overline{E} \times \overline{H}^* \right\} \quad [\text{W/m}^2]$$

Electrostatic Fields:

When there is no time-dependence of the fields, the electric and magnetic fields can exist as independent static fields. The electric fields are governed by

$$\nabla \cdot \overline{E} = \rho/\epsilon \qquad \oint_S \overline{E} \cdot d\overline{S} = \frac{1}{\epsilon} \int_V \rho \, dV = Q_{\text{enclosed}}/\epsilon \qquad \text{Gauss' law}$$

$$\nabla \times \overline{E} = 0 \qquad \oint_C \overline{E} \cdot d\ell = 0$$

Gauss' law by itself is not sufficient to solve problems, unless there is a high degree of symmetry that argues for only a single vector component of the field. Since $\nabla \times \overline{E} = 0$ the electric fields can be written as the gradient of a scalar potential,

$$\overline{E} = -\nabla V \qquad \text{or} \qquad V_a - V_b = -\int_b^a \overline{E} \cdot d\ell$$

From Gauss' law, this potential must satisfy Poisson's equation

$$\text{Poisson's Equation:} \qquad \nabla^2 V = -\rho/\epsilon$$

In charge free regions, this reduces to Laplace's equation

$$\text{Laplaces Equation:} \qquad \nabla^2 V = 0$$

Surfaces of constant potential are called equipotential surfaces, and are found to intersect the electric field at right angles. Perfect conducting surfaces are equipotential surfaces.

The force existing between two charges q_1 and q_2 has been found to obey Coulomb's Law:

$$\text{Coulomb's Law:} \qquad \overline{F} = \frac{q_1 q_2}{4\pi\epsilon r^2}\hat{r}$$

where r is the distance between charges, and ϵ is the permittivity of the intervening material. Using the principle of superposition, the field and potential due to an arbitrary charge distribution can be written as

$$\overline{E}(\overline{r}) = \int_V \frac{\rho(\overline{r}')}{4\pi\epsilon R^2}\hat{R}\,dV' \qquad V = \int_V \frac{\rho(\overline{r}')}{4\pi\epsilon R}\,dV'$$

where $R = |\overline{r} - \overline{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

In electrostatics, charge-storage devices can be created using multiple conductors separated by a dielectric. The capacitance and energy storage of these structures is given by

$$C = \frac{Q}{V} \qquad W_e = \frac{1}{2}CV^2$$

Electrostatic forces are computed as the gradient of the energy function,

$$F = \pm\nabla W_e \qquad \text{where: } \begin{cases} + & \text{used for fixed potential} \\ - & \text{used for fixed charge} \end{cases}$$

Magnetostatic Fields:

Static magnetic fields are governed by

$$\nabla \cdot \overline{H} = 0 \qquad \oint_S \overline{H} \cdot d\overline{S} = 0 \qquad \text{No Magnetic Charges}$$

$$\nabla \times \overline{H} = \overline{J} \qquad \oint_C \overline{H} \cdot d\ell = \int_S \overline{J} \cdot d\overline{S} = I_{\text{enclosed}} \qquad \text{Ampere's Law}$$

Magnetic fields are produced by steady (DC) currents. As in electrostatics, Ampere's law by itself is not sufficient to solve problems, unless there is a high degree of symmetry that argues for only a single vector component of the field.

The Biot-Savart law tells us how to compute the field for an arbitrary current distribution

$$\overline{H}(\overline{r}) = \int_\ell \frac{I d\ell \times \hat{R}}{4\pi R^2} dV' \qquad \text{or} \qquad \overline{H}(\overline{r}) = \int_V \frac{\overline{J}(\overline{r}') \times \hat{R}}{4\pi R^2} dV'$$

An important result in magnetostatics is the field of a long wire,

$$\overline{B} = \hat{\phi}\frac{\mu I}{2\pi r}$$

A wire carrying a DC current experiences a force in an applied field given by

$$\overline{F} = I \oint_C d\ell \times \overline{B}$$

When the wire is straight, this reduces to a force per unit length of IB, with the force at right angles to both the current and field.

The proportionality factor between total magnetic flux through a circuit and the source current is called the *inductance*. For a circuit with N loops, the inductance and stored energy is given by

$$L = \frac{\text{total flux linkage}}{\text{current}} = \frac{N}{I}\int_S \overline{B} \cdot d\overline{S} \qquad W_m = \frac{1}{2}LI^2$$

Since $\overline{B}$ should also be proportional to N, inductance usually varies as N^2. Magnetostatic forces are computed as the gradient of the energy function,

$$F = \pm\nabla W_m \qquad \text{where: } \begin{cases} + & \text{used for constant currents} \\ - & \text{used for constant flux linkage} \end{cases}$$

Time-Varying Fields and Phasors:

Maxwell's equations apply quite generally to all time varying electromagnetic phenomenon. However, time-harmonic, or sinusoidally varying fields are most commonly used, with time-dependence $\cos \omega t$. Although physically meaningful quantities can only be represented by real numbers, it is usually more convenient to introduce the complex exponential $e^{\jmath\omega t}$ rather than use real sinusoidal functions directly. Field quantities calculated with the complex exponential will generally be complex, and are called phasors. When phasors are used, the actual, or physically meaningful time-dependence can be recovered by using:

$$\overline{E}(\overline{r}, t) = \text{Re}\left\{\overline{E}(\overline{r})e^{\jmath\omega t}\right\}$$

where Re {} denotes the real part of what is in the brackets. In this form, the $\cos \omega t$ is taken as the reference for time-phase. For example, if the phasor field $\overline{E}(\overline{r})$ has a phase of 90°, the time dependence would be $\sin \omega t$. Using the $e^{\jmath\omega t}$ time dependence in Maxwell's equations allow us to eliminate the time variable completely, giving the phasor-form, or time-harmonic form of Maxwell's equations:

$$\nabla \times \overline{E} = -\jmath\omega\overline{B}$$
$$\nabla \times \overline{H} = \overline{J} + \jmath\omega\overline{D}$$
$$\nabla \cdot \overline{B} = 0$$
$$\nabla \cdot \overline{D} = \rho$$
$$\nabla \cdot \overline{J} = -\jmath\omega\rho$$

Although the same notation is used, the field variables in this form are now all phasor quantities.

Physical Constants:

Vacuum permittivity	$\epsilon_0 = 8.854 \times 10^{-12}$ F/m
Vacuum permeability	$\mu_0 = 4\pi \times 10^{-7}$ H/m
Electron charge magnitude	$q = 1.6022 \times 10^{-19}$ C
Electron rest mass	$m_0 = 9.1095 \times 10^{-31}$ kg
Speed of light in vacuum	$c = 2.997925 \times 10^8$ m/s

Commonly Used Greek Letters

alpha	α		iota	ι		rho	ρ	
beta	β		kappa	κ		sigma	σ	Σ
gamma	γ	Γ	lambda	λ	Λ	tau	τ	
delta	δ	Δ	mu	μ		upsilon	υ	Υ
epsilon	ϵ		nu	ν		phi	ϕ	Φ
zeta	ζ		xi	ξ	Ξ	chi	χ	
eta	η		omicron	o		psi	ψ	Ψ
theta	θ	Θ	pi	π	Π	omega	ω	Ω